THE SCHRÖDINGER EQUATION WITH A MOVING POINT INTERACTION IN THREE DIMENSIONS

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Abstract. In the case of a single point interaction we improve, by using different methods, the existence theorem for the unitary evolution generated by a Schrödinger operator with moving point interactions obtained by Dell'Antonio, Figari and Teta.

1. Introduction

Let us denote by $L^2(\mathbb{R}^3)$, with the usual scalar product $\langle\cdot,\cdot\rangle_2$ and corresponding norm $\|\cdot\|_2$, the Hilbert space of square integrable measurable functions on $\mathbb{R}^3$. By $H^1(\mathbb{R}^3)$ and by $H^2(\mathbb{R}^3)$ we denote the usual Sobolev-Hilbert spaces

$$H^1(\mathbb{R}^3) := \left\{\psi \in L^2(\mathbb{R}^3)\,:\,\nabla\psi \in L^2(\mathbb{R}^3)\right\}\,,$$

$$H^2(\mathbb{R}^3) := \left\{\psi \in H^1(\mathbb{R}^3)\,:\,\Delta\psi \in L^2(\mathbb{R}^3)\right\}\,.$$

Let

$$H \equiv -\Delta : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

be the self-adjoint operator giving the Hamiltonian of a free quantum particle in $\mathbb{R}^3$. For any $y \in \mathbb{R}^3$ let us consider the symmetric operator H_y° obtained by restricting H to the subspace

$$\{\psi \in H^2(\mathbb{R}^3)\,:\,\psi(y) = 0\}\,.$$

Such a symmetric operator has defect indices $(1,1)$. Any of its self-adjoint extensions different from H itself describes a point interaction centered at y. One has the following (see [1], section I.1.1 as regards $H_{\alpha,y}$ and see [9] as regards $F_{\alpha,y}$)

Theorem 1.1. *Any self-adjoint extension of H_y° different from H itself is given by*

$$H_{\alpha,y} : D(H_{\alpha,y}) \to L^2(\mathbb{R}^3)\,,$$

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$D(H_{\alpha,y}) :=$
$\{\psi \in L^2(\mathbb{R}^3) \,:\, \psi(x) = \psi_\lambda(x) + \Gamma_\alpha(\lambda)^{-1}\psi_\lambda(y)\,\mathcal{G}_\lambda(x-y)\,,\ \psi_\lambda \in H^2(\mathbb{R}^3)\}\,,$

$$(H_{\alpha,y} + \lambda)\psi := (H + \lambda)\psi_\lambda\,,$$

the definition being λ*-independent. Here* $\alpha \in \mathbb{R}$*,*

$$\Gamma_\alpha(\lambda) = \alpha + \frac{\sqrt{\lambda}}{4\pi}\,, \qquad \mathcal{G}_\lambda(x) = \frac{e^{-\sqrt{\lambda}\,|x|}}{4\pi|x|}$$

and $\lambda > 0$ *is chosen in such a way that* $\Gamma_\alpha(\lambda) \neq 0$*. The kernel of the resolvent of* $H_{\alpha,y}$ *is given by*

$$(H_{\alpha,y} + \lambda)^{-1}(x_1, x_2) = \mathcal{G}_\lambda(x_1 - x_2) + \Gamma_\alpha(\lambda)^{-1}\mathcal{G}_\lambda(x_1 - y)\,\mathcal{G}_\lambda(x_2 - y)\,.$$

The quadratic form associated with $H_{\alpha,y}$ *is*

$$F_{\alpha,y} : D(F_y) \times D(F_y) \to \mathbb{R}\,,$$

$D(F_y) :=$
$\left\{\psi \in H^1(\mathbb{R}^3) \,:\, \psi(x) = \psi_\lambda(x) + q_\psi\mathcal{G}_\lambda(x-y)\,,\ \psi_\lambda \in H^1(\mathbb{R}^3)\,,\ q_\psi \in \mathbb{C}\right\}\,,$

$$(F_{\alpha,y} + \lambda)(\psi,\phi) = \langle\nabla\psi_\lambda, \nabla\phi_\lambda\rangle_2 + \lambda\langle\psi_\lambda, \phi_\lambda\rangle_2 + \Gamma_\alpha(\lambda)\,\bar{q}_\psi q_\phi\,.$$

Moreover the essential spectrum of $H_{\alpha,y}$ *is purely absolutely continuous,*

$$\sigma_{ess}(H_{\alpha,y}) = \sigma_{ac}(H_{\alpha,y}) = [0,\infty)\,,$$
$$\alpha < 0 \quad \Rightarrow \quad \sigma_{pp}(H_{\alpha,y}) = -(4\pi\alpha)^2\,,$$
$$\alpha \geq 0 \quad \Rightarrow \quad \sigma_{pp}(H_{\alpha,y}) = \emptyset\,.$$

Suppose now that the point y is not fixed but describes a curve $y : \mathbb{R} \to \mathbb{R}^3$, thus producing the family of self-adjoint operators $H_{\alpha,y}(t) \equiv H_{\alpha,y(t)}$. Then one is interested in showing that the time-dependent Hamiltonian $H_{\alpha,y}(t)$ generates a strongly continuous unitary propagator $U_{t,s}$. Note that both the operator and the form domain of $H_{\alpha,y}(t)$ are strongly time-dependent. This renders inapplicable the known general theorems (see [3], [6]) and such a generation problem is not trivial.

By exploiting the explicit form of $H_{\alpha,y}(t)$ and in the case of several moving point interactions, Dell'Antonio, Figari and Teta obtained in [2] the following (here we state their results in the simpler case of a single point interaction)

Theorem 1.2. *Suppose that*

$$y \in C^3(\mathbb{R};\mathbb{R}^3)\,, \qquad \psi \in C_0^\infty(\mathbb{R}^3)\,, \qquad \psi(y(s)) = 0\,.$$

Then there exist an unique strongly continuous unitary propagator

$$U_{t,s} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

such that

$$\left(i\frac{d}{dt}\,U_{t,s}\psi,\phi\right)_t = F_{\alpha,y}(t)(U_{t,s}\psi,\phi) \tag{1.1}$$

for all $\phi \in D(F_y(t))$.

Here $(\cdot,\cdot)_t$ denotes the duality between $D(F_y(t))$ and its strong dual. Moreover the solution $\psi(t) := U_{t,s}\psi$ has a natural representation (see [2], equations (14)-(21) for the details).

In the introduction of [2] the authors conjectured that $U_{t,s}$ defines a flow on $D(F_y(t))$ which is continuous with respect to the Banach topology induced by the quadratic form $F_{\alpha,y}(t)$.

Here we show, by using different methods, that if $y \in C^2(\mathbb{R};\mathbb{R}^3)$ then this is indeed the case and Theorem 1.2 above holds for any $\psi \in D(F_y(s))$ (see Theorem 3.1 for the precise statements).

Our proof procedes in the following conceptually simple way. Noticing that the unitary map

$$T_t\psi(x) := \psi(x+y(t))$$

transforms the equation

$$i\,\frac{d\psi}{dt}(t) = H\psi(t)$$

into the nonautonomous one

$$i\,\frac{d\psi}{dt}(t) = H_{\mathsf{v}}(t)\psi(t) \equiv (H + i\mathsf{v}(t)\cdot\nabla)\,\psi(t)\,, \qquad \mathsf{v}(t) \equiv \frac{dy}{dt}(t)\,,$$

we consider the point perturbations (at $y=0$) of H_{v}, where v is a given, time-independent vector in $\mathbb{R}^3$. We realize (see Theorem 2.3) that the form domains $D(F_{\mathsf{v}})$ of such singular perturbations $H_{\mathsf{v},\alpha}$ of H_{v} are v-independent. Indeed one has $D(F_{\mathsf{v}}) \equiv D(F_0)$, where $D(F_0)$ is the form domain of $H_{\alpha,y}$, $y=0$. This allows, in the case the vector v is time-dependent, the application of Kisyński theorem (see the Appendix) thus obtaining a strongly continuous unitary propagator $\widetilde{U}_{t,s}$ which is also a strongly continuous propagator on $D(F_0)$ with respect to the Banach topology induced by the quadratic form associated with $H_{\alpha,y}$, $y=0$. Moreover $F_{\mathsf{v},\alpha}$, the quadratic form associated with $H_{\mathsf{v},\alpha}$, and $F_{\alpha,0}$, the quadratic form associated with $H_{\alpha,0}$, are related by the identity (see Theorem 2.3 again)

$$F_{\mathsf{v},\alpha} = F_{\alpha,0} + Q_{\mathsf{v}}\,,$$

where Q_{v} is the quadratic form associated with the natural extension of $i\mathsf{v}\cdot\nabla$ to $D(F_0)$ (see Remark 2.4). This allows us to show (see Theorem

3.1) that

$$U_{t,s} := T_t^{-1}\tilde{U}_{t,s}T_s$$

satisfies (1.1) for any $\psi \in D(F_{\alpha,y}(s))$ and is a continuous flow from $D(F_{\alpha,y}(s))$ onto $D(F_{\alpha,y}(t))$. In the case $y \in C^3(\mathbb{R};\mathbb{R}^3)$ we also show that $U_{t,s}$ maps $\tilde{D}(H_{\alpha,y}(s))$ onto $\tilde{D}(H_{\alpha,y}(t))$, where

$$\tilde{D}(H_{\alpha,y}(t)) := V_t D(H_{\alpha,y}(t))\,, \quad V_t\psi(x) := e^{i\mathsf{v}(t)\cdot x/2}\psi(x)\,.$$

2. The operator $-\Delta + iL_{\mathsf{v}}$ with a point interaction

Let us consider the linear operator $-\Delta + iL_{\mathsf{v}}$, where

$$L_{\mathsf{v}}\psi := \sum_{k=1}^{3}\mathsf{v}_k\partial_k\psi\,, \quad \mathsf{v} \equiv (\mathsf{v}_1,\mathsf{v}_2,\mathsf{v}_3) \in \mathbb{R}^3\,.$$

Since, for any $\epsilon > 0$,

$$\begin{aligned}\|L_{\mathsf{v}}\psi\|_2^2 \le &\sum_{1\le k,j\le 3}\left|\mathsf{v}_k\mathsf{v}_j\langle\partial^2_{kj}\psi,\psi\rangle_2\right| \le 3|\mathsf{v}|^2|\langle\Delta\psi,\psi\rangle_2|\\ &\le\frac{3}{2}\,|\mathsf{v}|^2\left(\epsilon\|\Delta\psi\|_2^2 + \frac{1}{\epsilon}\,\|\psi\|_2^2\right)\,,\end{aligned}$$

by Kato-Rellich theorem one has that

$$H_{\mathsf{v}} := -\Delta + iL_{\mathsf{v}} : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

is self-adjoint. Moreover, since, for any $\epsilon > 0$,

$$|\langle L_{\mathsf{v}}\psi,\psi\rangle_2| \le |\mathsf{v}|\,\|\nabla\psi\|_2\|\psi\|_2 \le \frac{|\mathsf{v}|}{2}\left(\epsilon\|\nabla\psi\|_2^2 + \frac{1}{\epsilon}\,\|\psi\|_2^2\right)\,,$$

H_{v} has lower bound $-|\mathsf{v}|^2/4$.

Now we look for the self-adjoint extensions of the symmetric operator H°_{v} obtained by restricting H_{v} to the kernel of the continuous and surjective linear map

$$\tau : H^2(\mathbb{R}^3) \to \mathbb{C}\,, \qquad \tau\psi := \psi(0)\,.$$

Theorem 2.1. *Any self-adjoint extension of H°_{v} different from H_{v} itself is given by*

$$H_{\mathsf{v},\alpha} : D(H_{\mathsf{v},\alpha}) \to L^2(\mathbb{R}^3)\,,$$

$$D(H_{\mathsf{v},\alpha}) := \left\{\psi \in L^2(\mathbb{R}^3)\,:\,\psi = \psi_\lambda + \Gamma_{\mathsf{v},\alpha}(\lambda)^{-1}\psi_\lambda(0)\,\mathcal{G}^{\mathsf{v}}_\lambda\,,\ \psi_\lambda \in H^2(\mathbb{R}^3)\right\}\,,$$

$$(H_{\mathsf{v},\alpha} + \lambda)\psi := (H_{\mathsf{v}} + \lambda)\psi_\lambda\,,$$

the definition being λ-independent. Here $\alpha \in \mathbb{R}$,

$$\Gamma_{\mathsf{v},\alpha}(\lambda) = \alpha + \frac{\sqrt{\lambda - |\mathsf{v}|^2/4}}{4\pi}\,, \qquad \mathcal{G}^{\mathsf{v}}_\lambda(x) = \frac{e^{-\sqrt{\lambda-|\mathsf{v}|^2/4}\,|x|}}{4\pi|x|}\,e^{i\mathsf{v}\cdot x/2}$$

and $\lambda > |\mathsf{v}|^2/4$ *is chosen in such a way that* $\Gamma_{\mathsf{v},\alpha}(\lambda) \neq 0$. *The kernel of the resolvent of* $H_{\mathsf{v},\alpha}$ *is given by*

$$(H_{\mathsf{v},\alpha} + \lambda)^{-1}(x_1, x_2) = \mathcal{G}^{\mathsf{v}}_\lambda(x_1 - x_2) + \Gamma_{\mathsf{v},\alpha}(\lambda)^{-1}\mathcal{G}^{\mathsf{v}}_\lambda(x_1)\,\mathcal{G}^{-\mathsf{v}}_\lambda(x_2)\,.$$

Moreover the spectrum of $H_{\mathsf{v},\alpha}$ *is purely absolutely continuous,*

$$\sigma_{ess}(H_{\mathsf{v},\alpha}) = \sigma_{ac}(H_{\mathsf{v},\alpha}) = [\,-|\mathsf{v}|^2/4, \infty)\,,$$

$$\alpha < 0 \quad \Rightarrow \quad \sigma_{pp}(H_{\mathsf{v},\alpha}) = -(4\pi\alpha)^2 - |\mathsf{v}|^2/4$$

$$\alpha \geq 0 \quad \Rightarrow \quad \sigma_{pp}(H_{\mathsf{v},\alpha}) = \emptyset\,.$$

Proof. Let us define the bounded linear operators

$$\breve{G}(\lambda) : L^2(\mathbb{R}^3) \to \mathbb{C}\,, \qquad \breve{G}(\lambda) := \tau(H_{\mathsf{v}} + \lambda)^{-1}$$

and

$$G(\lambda) : \mathbb{C} \to L^2(\mathbb{R}^3)\,, \qquad G(\lambda) := \breve{G}(\lambda)^*\,.$$

Since

$$\mathcal{G}^{\mathsf{v}}_\lambda(x) = \frac{e^{-\sqrt{\lambda - |\mathsf{v}|^2/4}\,|x|}}{4\pi|x|}\,e^{i\mathsf{v}\cdot x/2}$$

is the Green function of $H_{\mathsf{v}} + \lambda$ (see e.g. [5]), one obtains

$$\breve{G}(\lambda)\psi = \langle \mathcal{G}^{-\mathsf{v}}_\lambda, \psi\rangle_2\,, \qquad G(\lambda)q = q\,\mathcal{G}^{\mathsf{v}}_\lambda\,.$$

Since (see [7], Lemma 2.1)

$$(\mu - \lambda)(H_{\mathsf{v}} + \lambda)^{-1}G(\mu) = G(\lambda) - G(\mu)\,,$$

one obtains (see [7], Lemma 2.2)

$$\begin{aligned}
&(\mu - \lambda)\breve{G}(\lambda)G(\mu)\\
=&\tau(G(\lambda) - G(\mu)) = \tau(G(\nu) - G(\mu)) - \tau(G(\nu) - G(\lambda))\\
=&\frac{\sqrt{\mu - |\mathsf{v}|^2/4}}{4\pi} - \frac{\sqrt{\nu - |\mathsf{v}|^2/4}}{4\pi} - \left(\frac{\sqrt{\lambda - |\mathsf{v}|^2/4}}{4\pi} - \frac{\sqrt{\nu - |\mathsf{v}|^2/4}}{4\pi}\right)\\
=&\frac{\sqrt{\mu - |\mathsf{v}|^2/4}}{4\pi} - \frac{\sqrt{\lambda - |\mathsf{v}|^2/4}}{4\pi}\,.
\end{aligned}$$

The thesis about $H_{\mathsf{v},\alpha}$ and its resolvent then follows from Theorem 2.1 in [7]. As regards the spectral properties of $H_{\mathsf{v},\alpha}$ one procedes as in [1], Theorem 1.1.4 . □

Remark 2.2. Note that, as expected, $H_{\mathsf{v},\alpha}$ converges in norm resolvent sense to $H_{\alpha,0}$ as $|\mathsf{v}| \downarrow 0$.

Theorem 2.3. *The quadratic form associated with $H_{\mathsf{v},\alpha}$ is*

$$F_{\mathsf{v},\alpha} : D(F_0) \times D(F_0) \to \mathbb{R}\,, \qquad F_{\mathsf{v},\alpha} = F_{\alpha,0} + Q_{\mathsf{v}}\,,$$

where $D(F_0)$ is the domain of the quadratic form $F_{\alpha,0}$ associated with $H_{\alpha,y}$, $y = 0$, (see Theorem 1.1) and

$$Q_{\mathsf{v}} : D(F_0) \times D(F_0) \to \mathbb{R}$$

$$Q_{\mathsf{v}}(\psi, \phi) := \langle iL_{\mathsf{v}}\psi_\lambda, \phi_\lambda\rangle_2 + \bar q_\psi \langle \mathcal{G}_\lambda, iL_{\mathsf{v}}\phi_\lambda\rangle_2 + q_\phi \langle iL_{\mathsf{v}}\psi_\lambda, \mathcal{G}_\lambda\rangle_2\,.$$

Proof. Given ψ and ϕ in $D(H_{\mathsf{v},\alpha})$ put

$$q_\psi := \Gamma_{\mathsf{v},\alpha}(\lambda)^{-1}\psi_\lambda(0)\,, \qquad q_\phi := \Gamma_{\mathsf{v},\alpha}(\lambda)^{-1}\phi_\lambda(0)\,.$$

Then

$$\begin{aligned}\langle (H_{\mathsf{v},\alpha} + \lambda)\psi, \phi\rangle_2 =& \langle (H_{\mathsf{v}} + \lambda)\psi_\lambda, \phi_\lambda\rangle_2 + q_\phi \langle (H_{\mathsf{v}} + \lambda)\psi_\lambda, \mathcal{G}^{\mathsf{v}}_\lambda\rangle_2\\ =& \langle (H_{\mathsf{v}} + \lambda)\psi_\lambda, \phi_\lambda\rangle_2 + \Gamma_{\mathsf{v},\alpha}(\lambda)\,\bar q_\psi q_\phi\,.\end{aligned}$$

Thus one is lead to define the quadratic form

$$F_{\mathsf{v},\alpha} : D(F_{\mathsf{v}}) \times D(F_{\mathsf{v}}) \to \mathbb{R}$$

by

$$D(F_{\mathsf{v}}) := \left\{\psi \in H^1(\mathbb{R}^3)\,:\, \psi = \psi_\lambda + q_\psi \mathcal{G}^{\mathsf{v}}_\lambda\,, \quad \psi_\lambda \in H^1(\mathbb{R}^3)\,,\ q_\psi \in \mathbb{C}\right\}\,,$$

$$\begin{aligned}&(F_{\mathsf{v},\alpha} + \lambda)(\psi, \phi)\\ :=& \langle(-\Delta + iL_{\mathsf{v}} + \lambda)^{1/2}\psi_\lambda, (-\Delta + iL_{\mathsf{v}} + \lambda)^{1/2}\phi_\lambda\rangle_2 + \Gamma_{\mathsf{v},\alpha}(\lambda)\,\bar q_\psi q_\phi\,.\end{aligned}$$

It is then straightforward to check that $F_{\mathsf{v},\alpha}$ is closed and bounded from below. Thus $F_{\mathsf{v},\alpha}$ is the quadratic form associated with $H_{\mathsf{v},\alpha}$. Since

$$\mathcal{G}^{\mathsf{v}}_\lambda - \mathcal{G}_\lambda \in H^1(\mathbb{R}^3)$$

one obtains $D(F_{\mathsf{v}}) \equiv D(F_0)$. Re-writing the quadratic form above by using the decomposition entering in the definition of $D(F_0)$ and noticing that

$$\forall \psi \in H^1(\mathbb{R}^3) \qquad (F_{\mathsf{v}} + \lambda)(\mathcal{G}^{\mathsf{v}}_\lambda - \mathcal{G}_\lambda, \psi) = \langle \mathcal{G}_\lambda, iL_{\mathsf{v}}\psi\rangle_2$$

one obtains

$$\begin{aligned}&(F_{\mathsf{v},\alpha} + \lambda)(\psi, \phi) = \langle \nabla\psi_\lambda, \nabla\phi_\lambda\rangle_2 + \lambda\langle \psi_\lambda, \phi_\lambda\rangle_2 + Q_{\mathsf{v}}(\psi, \phi)\\ &+ (\Gamma_{\mathsf{v},\alpha}(\lambda) + (F_{\mathsf{v},\alpha} + \lambda)(\mathcal{G}^{\mathsf{v}}_\lambda - \mathcal{G}_\lambda, \mathcal{G}^{\mathsf{v}}_\lambda - \mathcal{G}_\lambda))\,\bar q_\psi q_\phi\,,\end{aligned}$$

Since L_{v} is skew-adjoint one has $\langle L_{\mathsf{v}}\psi, \psi\rangle_2 = 0$ for any real-valued $\psi \in H^1(\mathbb{R}^3)$. Thus, by taking a real-valued $J_\epsilon \in C^\infty_0(\mathbb{R}^3)$ such that J_ϵ

weakly converges to the Dirac mass at the origin as $\epsilon\downarrow 0$, one obtains (here $*$ denotes convolution)

$$\begin{aligned}
&(F_{\mathsf{v},\alpha}+\lambda)(\mathcal{G}^{\mathsf{v}}_\lambda-\mathcal{G}_\lambda,\mathcal{G}^{\mathsf{v}}_\lambda-\mathcal{G}_\lambda)=\langle\mathcal{G}_\lambda,iL_{\mathsf{v}}(\mathcal{G}^{\mathsf{v}}_\lambda-\mathcal{G}_\lambda)\rangle_2\\
=&\lim_{\epsilon\downarrow 0}\langle iL_{\mathsf{v}}\mathcal{G}_\lambda*J_\epsilon,\mathcal{G}^{\mathsf{v}}_\lambda-\mathcal{G}_\lambda*J_\epsilon\rangle_2=\lim_{\epsilon\downarrow 0}\langle iL_{\mathsf{v}}\mathcal{G}_\lambda*J_\epsilon,\mathcal{G}^{\mathsf{v}}_\lambda\rangle_2=\\
&=\lim_{\epsilon\downarrow 0}\langle(H_{\mathsf{v}}+\lambda)(\mathcal{G}^{\mathsf{v}}_\lambda-\mathcal{G}_\lambda)*J_\epsilon,\mathcal{G}^{\mathsf{v}}_\lambda\rangle_2\\
=&\lim_{\epsilon\downarrow 0}\langle\mathcal{G}^{\mathsf{v}}_\lambda-\mathcal{G}_\lambda,(H_{\mathsf{v}}+\lambda)G^{\mathsf{v}}_\lambda*J_\epsilon,\rangle_2=\lim_{\epsilon\downarrow 0}\langle\mathcal{G}^{\mathsf{v}}_\lambda-\mathcal{G}_\lambda,J_\epsilon,\rangle_2=\\
=&(\mathcal{G}^{\mathsf{v}}_\lambda-\mathcal{G}_\lambda)(0)=\frac{\sqrt{\lambda}-\sqrt{\lambda-|\mathsf{v}|^2/4}}{4\pi}\equiv\Gamma_\alpha(\lambda)-\Gamma_{\mathsf{v},\alpha}(\lambda)
\end{aligned}$$

and the proof is done. □

Remark 2.4. Let J_ϵ be a real-valued, compactly supported smooth function weaky converging to the Dirac mass a zero as $\epsilon\downarrow 0$. For any $\psi=\psi_\lambda+q_\psi\mathcal{G}_\lambda$ and $\phi=\phi_\lambda+q_\phi\mathcal{G}_\lambda$, let us define $\psi_\epsilon:=\psi_\lambda+q_\psi\mathcal{G}_\lambda*J_\epsilon$ and $\phi_\epsilon:=\phi_\lambda+q_\phi\mathcal{G}_\lambda*J_\epsilon$. Then, since L_{v} is skew-adjoint, one has

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\langle iL_{\mathsf{v}}\psi_\epsilon,\phi_\epsilon\rangle_2=\lim_{\epsilon\downarrow 0}\,(\langle iL_{\mathsf{v}}\psi_\lambda,\phi_\lambda\rangle_2+\bar q_\psi\langle\mathcal{G}_\lambda*J_\epsilon,iL_{\mathsf{v}}\phi_\lambda\rangle_2\\
&+q_\phi\langle iL_{\mathsf{v}}\psi_\lambda,\mathcal{G}_\lambda*J_\epsilon\rangle_2-i\bar q_\psi q_\phi\langle L_{\mathsf{v}}\mathcal{G}_\lambda*J_\epsilon,\mathcal{G}_\lambda*J_\epsilon\rangle_2)\\
=&\lim_{\epsilon\downarrow 0}\,Q_{\mathsf{v}}(\psi_\epsilon,\phi_\epsilon)=Q_{\mathsf{v}}(\psi,\phi)\,.
\end{aligned}$$

Thus Q_{v} is the natural extension to $D(F_0)$ of the quadratic form associated with iL_{v}.

3. The Schrödinger equation with a moving point interaction

Let us now consider a differentiable curve $y:\mathbb{R}\to\mathbb{R}^3$ and put $\mathsf{v}(t)\equiv\frac{dy}{dt}(t)$. Thus one has the families of self-adjoint operators and associated quadratic forms

$$\begin{gathered}
H_{\alpha,y}(t):D(H_{\alpha,y}(t))\to L^2(\mathbb{R}^3)\,,\\
F_{\alpha,y}(t):D(F_y(t))\times D(F_y(t))\to\mathbb{R}\,,\\
H_{\mathsf{v},\alpha}(t):D(H_{\alpha,y}(t))\to L^2(\mathbb{R}^3)\,,\\
F_{\mathsf{v},\alpha}(t):D(F_0)\times D(F_0)\to\mathbb{R}\,.
\end{gathered}$$

Now we can state our main result:

Theorem 3.1. *Let $y\in C^2(\mathbb{R};\mathbb{R}^3)$. Then there is a unique strongly continuous unitary propagator*

$$U_{t,s}:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)\,,\qquad (t,s)\in\mathbb{R}^2\,,$$

such that

1)

$$U_{t,s}D(F_{\alpha,y}(s)) = D(F_{\alpha,y}(t))\,;$$

2) each $U_{t,s}$ *is strongly continuous as a map from* $D(F_y(s))$ *onto* $D(F_y(t))$ *with respect to the Banach topologies induced by the bounded from below closed quadratic forms* $F_{\alpha,y}(s)$ *and* $F_{\alpha,y}(t)$ *respectively;*

3)

$$\forall\psi\in D(F_y(s))\,,\qquad t\mapsto F_{\alpha,y}(t)(U_{t,s}\psi,U_{t,s}\psi)\qquad \textit{is in}\quad C(\mathbb{R};\mathbb{R})\,;$$

4)

$$\forall\psi\in D(F_y(s))\,,\qquad t\mapsto U_{t,s}\psi\qquad \textit{is in}\quad C^1\left(\mathbb{R};D(F_y(\cdot))^*\right)\,,$$

where $D(F_y(t))^*$ *denotes the dual of* $D(F_y(t))$ *with respect to the* $L^2(\mathbb{R}^3)$ *scalar product;*

5)

$$\forall\psi\in D(F_y(s))\,,\,\forall\phi\in D(F_y(t))\,,\quad \left(i\,\frac{d}{dt}\,U_{t,s}\psi,\phi\right)_t = F_{\alpha,y}(t)(U_{t,s}\psi,\phi)\,,$$

where $(\cdot,\cdot)_t$ *denotes the duality between* $D(F_y(t))$ *and* $D(F_y(t))^*$.
If $y\in C^3(\mathbb{R};\mathbb{R}^3)$ *then*

6)

$$U_{t,s}\tilde D(H_{\alpha,y}(s)) = \tilde D(H_{\alpha,y}(t))\,,$$

where

$$\tilde D(H_{\alpha,y}(t)) := V_t D(H_{\alpha,y}(t))\,,\quad V_t\psi(x) := e^{i\mathsf{v}(t)\cdot x/2}\psi(x)\,.$$

Proof. By Theorem 2.3 we have that $y\in C^2(\mathbb{R};\mathbb{R}^3)$ implies that

$$\forall\,\psi,\phi\in D(F_0)\,,\qquad t\mapsto F_{\mathsf{v},\alpha}(t)(\psi,\phi)\qquad \text{is in}\quad C^1(\mathbb{R})\,.$$

Let $T>0$. By Kisyński's theorem (see the Appendix) applied to the family of strictly positive self-adjoint operators

$$H_{\mathsf{v},\alpha}(t)+\lambda\,,\quad t\in[-T,T]\,,\quad \lambda>(4\pi\min(0,\alpha))^2+\frac{1}{4}\sup_{t\in[-T,T]}|\mathsf{v}(t)|\,,$$

one knows that $H_{\mathsf{v},\alpha}(t)$, $t\in[-T,T]$, generates a strongly continuous unitary propagator $\tilde U^T_{t,s}$, $(s,t)\in[-T,T]^2$. By unicity if $T'>T$ then $\tilde U^T_{s,t} = U^{T'}_{s,t}$ for any $(s,t)\in[-T,T]^2\subset[-T',T']^2$. Thus we obtain an unique strongly continuous unitary propagator $\tilde U_{t,s}$, $(s,t)\in\mathbb{R}^2$, generated by the family $H_{\mathsf{v},\alpha}(t)$, $t\in\mathbb{R}$. Such a propagator is also a strongly continuous propagator on $D(F_0)$ with respect to the Banach topology induced by the bounded from below closed quadratic form $F_{\alpha,0}$.

Considering the unitary map

$$T_t: L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)\,,\qquad T_t\psi(x) := \psi(x+y(t))\,,$$

we define then the strongly continuous unitary propagator

$$U_{t,s} := T_t^{-1}\tilde{U}_{t,s}T_s\,.$$

Since T_t is a bounded operator from $D(F_y(t))$ onto $D(F_0)$ one has that $U_{t,s}$ is a bounded operator from $D(F_y(s))$ onto $D(F_y(t))$ with respect to the Banach topologies induced by the bounded from below closed quadratic forms $F_{\alpha,y}(s)$ and $F_{\alpha,y}(t)$ respectively. Moreover, for all $\psi \in D(F_y(s))$, the map

$$t \mapsto F_{\alpha,y}(t)(U_{t,s}\psi, U_{t,s}\psi) \equiv F_{\alpha,0}(\tilde{U}_{t,s}T_s\psi, \tilde{U}_{t,s}T_s\psi)$$

is continuous. Let us now show that, for all $\psi \in D(F_y(s))$ and for all $\phi \in D(F_y(t))$ one has

$$\left(i\,\frac{d}{dt}\,U_{t,s}\psi, \phi\right)_t = F_{\alpha,y}(t)(U_{t,s}\psi, \phi)\,.$$

For any $\psi \in D(F_y(s))$ and $\phi \in D(F_y(t))$ there exist $\tilde{\psi}$ and $\tilde{\phi}$ in $D(F_0)$ such that $T_s^{-1}\tilde{\psi} = \psi$ and $T_t^{-1}\tilde{\phi} = \phi$. Thus equivalently we need to show that

$$\left(i\,\frac{d}{dt}\,T_t^{-1}\tilde{U}_{t,s}\tilde{\psi}, T_t^{-1}\tilde{\phi}\right)_t = F_{\alpha,y}(t)(T_t^{-1}\tilde{U}_{t,s}\tilde{\psi}, T_t^{-1}\tilde{\phi}) \equiv F_{\alpha,0}(\tilde{U}_{t,s}\tilde{\psi}, \tilde{\phi})\,.$$

Since

$$\begin{aligned}\left(i\,\frac{d}{dt}\,T_t^{-1}\tilde{U}_{t,s}\tilde{\psi}, T_t^{-1}\tilde{\phi}\right)_t &= \left(i\,T_t\,\frac{d}{dt}\,T_t^{-1}\tilde{U}_{t,s}\tilde{\psi}, \tilde{\phi}\right)\\ =\left(i\,T_t\left(\frac{d}{dt}\,T_t^{-1}\right)\tilde{U}_{t,s}\tilde{\psi}, \tilde{\phi}\right) &+ \left(i\,\frac{d}{dt}\,\tilde{U}_{t,s}\tilde{\psi}, \tilde{\phi}\right)\\ =\left(i\,T_t\left(\frac{d}{dt}\,T_t^{-1}\right)\tilde{U}_{t,s}\tilde{\psi}, \tilde{\phi}\right) &+ F_{\mathsf{v},\alpha}(t)(\tilde{U}_{t,s}\tilde{\psi}, \tilde{\phi})\,,\end{aligned}$$

by Theorem 2.3 we need to show that

$$\left(i\,T_t\,\frac{d}{dt}\,T_t^{-1}\tilde{\psi}, \tilde{\phi}\right) = -Q_{\mathsf{v}}(\tilde{\psi}, \tilde{\phi})\,.$$

This is obviously true in the case either $\tilde{\psi}$ or $\tilde{\phi}$ is in $H^1(\mathbb{R}^3)$ and, by taking J_ϵ as in Remark 2.4,

$$\begin{aligned}\left(T_t\,\frac{d}{dt}\,T_t^{-1}\mathcal{G}_\lambda, \mathcal{G}_\lambda\right) &= \lim_{\epsilon\downarrow 0}\,\left\langle T_t\,\frac{d}{dt}\,T_t^{-1}\mathcal{G}_\lambda * J_\epsilon, \mathcal{G}_\lambda * J_\epsilon\right\rangle_2\\ = -\lim_{\epsilon\downarrow 0}\,\langle L_{\mathsf{v}}\mathcal{G}_\lambda * J_\epsilon, \mathcal{G}_\lambda * J_\epsilon\rangle_2 &= 0\,.\end{aligned}$$

Thus point 5) is proven. Point 6) follows from Kisyński's theorem again by noticing that if $y \in C^3(\mathbb{R};\mathbb{R}^3)$ then $\tilde{U}_{t,s}$ maps $D(H_{\mathsf{v},\alpha}(s))$ onto $D(H_{\mathsf{v},\alpha}(t))$ and that

$$D(H_{\mathsf{v},\alpha}(t)) \equiv T_t V_t D(H_{\alpha,y}(t))\,.$$

□

4. Appendix: the Kisyński's theorem

For the reader's convenience in this appendix we recall Kisyński's theorem. For the proof we refer to Kisyński's original paper [4] (see in particular [4], section 8. Also see [8], section II.7).

Let us remind that the double family $U_{t,s}$, $(t,s) \in [T_1,T_2]^2$, is said to be a strongly continuous propagator on the Hilbert space $\mathscr{H}$ if each $U_{t,s}$ is a bounded operator on $\mathscr{H}$, the map $(t,s) \mapsto U_{t,s}$ is strongly continuous, $U_{s,s} = 1$ and the Chapman-Kolmogorov equation

$$U_{t,r}U_{r,s} = U_{t,s}$$

holds. Such a propagator is then said to be unitary if each $U_{t,s}$ is unitary.

Theorem 4.1. *Let $A(t)$, $t \in [T_1,T_2]$, be a family of strictly positive self-adjoint operators on the Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ with time-independent form domain $\mathscr{H}_+$. Suppose that*

$$\forall \psi,\phi \in \mathscr{H}_+ \qquad t \mapsto F(t)(\psi,\phi) \qquad \textit{is in} \quad C^k\left([T_1,T_2];\mathbb{R}\right)\,,$$

where $F(t)$ denotes the quadratic form associated with $A(t)$.
If $k=1$ then there is a unique strongly continuous unitary propagator

$$U_{t,s} : \mathscr{H} \to \mathscr{H}\,, \qquad (s,t) \in [T_1,T_2]^2\,,$$

such that
1)

$$U_{t,s}\mathscr{H}_+ = \mathscr{H}_+\,;$$

2) $U_{t,s}$ is a strongly continuous propagator on $(\mathscr{H}_+, \langle\cdot,\cdot\rangle_+)$, where $\langle\cdot,\cdot\rangle_+$ denotes any of the equivalent scalar products

$$\langle\psi,\phi\rangle_{t,+} := F(t)(\psi,\phi)\,;$$

3)

$$\forall \psi \in \mathscr{H}_+ \qquad t \mapsto U_{t,s}\psi \qquad \textit{is in} \quad C^1\left([T_1,T_2];\mathscr{H}_-\right)\,,$$

where $(\mathscr{H}_-, \langle\cdot,\cdot\rangle_-)$ is the completion of $\mathscr{H}$ endowed with any of the equivalent scalar products

$$\langle\psi,\phi\rangle_{t,-} := \langle A(t)^{-1/2}\psi, A(t)^{-1/2}\phi\rangle\,;$$

4)

$$\forall\psi,\phi\in\mathscr{H}_+\qquad \left(i\,\frac{d}{dt}\,U_{t,s}\psi,\phi\right)=F(t)(U_{t,s}\psi,\phi)\,,$$

where $(\cdot,\cdot)$ *denotes the duality between* $\mathscr{H}_+$ *and* $\mathscr{H}_-$*. If* $k=2$ *then*

5)

$$U_{t,s}\,D(A(s))=D(A(t))\,,$$

6)

$$\forall\psi\in D(A(s))\quad t\mapsto U_{t,s}\psi\quad \textit{is in}\quad C^1([T_1,T_2];\mathscr{H})\cap C([T_1,T_2];D(A(\cdot)))\,,$$

7)

$$i\,\frac{d}{dt}\,U_{t,s}\psi=A(t)U_{t,s}\psi\,.$$

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